Lecture Notes for Chapter 2

International Financial Markets and Institutions

Chapter 2

Preliminaries: Conventions, notation, and basic concepts

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Road Map

12 Summary of international finance

13 Revision classes

2.15 Black-Scholes Again!

Remember that the payoff for a call option with strike K is

$$\max(\tilde{S}_T - K, 0). \tag{2.1}$$

This is

- a future value
- risky – to emphasize this we place a $\tilde{\cdot}$ over the price the stock at date T

To find the present value (at date t) of this future value we

- Compute the certainty equivalent (i.e. expectation using risk-neutral probabilities) of the future value.

$$CEQ_t[\max(\tilde{S}_T - K, 0)] \tag{2.2}$$

  - By using risk-neutral probabilities, we have already priced the systematic risk in the future value

- Discount at the risk-free rate

$$\frac{1}{1 + r_{t,T}} CEQ_t[\max(\tilde{S}_T - K, 0)] \tag{2.3}$$

Some care is needed with the risk-free discount factor $\frac{1}{1+r_{t,T}}$. In text books, you are more likely to see $e^{-r(T-t)}$. This is because

- Risk-free rates are assumed to be constant
- $r \times (T - t)$ is just the continuously compounded rate over the time interval $[t, T]$
- $r_{t,T}$ is the annually compounded rate over the time interval $[t, T]$
- The two are linked together via

$$e^{-r(T-t)} = \frac{1}{1 + r_{t,T}} \tag{2.4}$$

Text books: common to replace CEQ_t via $E_t^{\mathbb{Q}}$ – superscript $\mathbb{Q}$ reminds us to use risk-neutral probabilities

$$\frac{1}{1+r_{t,T}} CEQ_t[\max(\tilde{S}_T - K, 0)] = e^{-r(T-t)} E_t^{\mathbb{Q}}[\max(\tilde{S}_T - K, 0)] \tag{2.5}$$

To evaluate $E_t^{\mathbb{Q}}[\max(\tilde{S}_T - K, 0)]$, need to know risk-neutral probability distribution for the stock price at date T

How can we make this easy?

Make following easy assumption:

- $\tilde{S}_T$ is equal to S_u with risk-neutral probability q and S_d with risk-neutral probability $1 - q$, this is a one period binomal tree!
- We then have

$$E_t^{\mathbb{Q}}[\max(\tilde{S}_T - K, 0)] = q \max(S_u - K, 0) \tag{2.6}$$
$$+ (1 - q) \max(S_d - K, 0) \tag{2.7}$$

The Black-Scholes analysis makes a harder but more realistic assumption about the risk-neutral probability distribution

$$\ln \frac{\tilde{S}_T}{S_t} \sim N\left[\left(r - \frac{1}{2}\sigma^2\right)(T-t), \sigma\sqrt{T-t}\right] \tag{2.8}$$

We then have

$$E_t^{\mathbb{Q}}[\max(\tilde{S}_T - K, 0)] = e^{r(T-t)}N(d_{1,t}) - KN(d_{2,t}) \tag{2.9}$$

Hence

$$e^{-r(T-t)}E_t^{\mathbb{Q}}[\max(\tilde{S}_T - K, 0)] = N(d_{1,t})S_t - e^{-r(T-t)}KN(d_{2,t}) \tag{2.10}$$

Heuristic interpretation

- $N(d_{2,t})$ is the risk-neutral probability that $\tilde{S}_T > K$ (option is in money at expiry)
- $N(d_{1,t}) \approx N(d_{2,t})$

What happens when $N(d_{1,t}) = N(d_{2,t}) = 1$?

$$
\begin{aligned}
e^{-r(T-t)} E_t^{\mathbb{Q}}[\max(\tilde{S}_T - K, 0)] &= N(d_{1,t}) S_t - e^{-r(T-t)} K N(d_{2,t}) &\quad (2.11)\\
&= \underbrace{S_t}_{PV_t[\tilde{S}_T]} - \underbrace{e^{-r(T-t)} K}_{PV_t[K]} &\quad (2.12)
\end{aligned}
$$

More precise interpretation (non-examinable)

$$\max(\tilde{S}_T - K, 0) = \tilde{S}_T 1_{\{\tilde{S}_T > K\}} - K 1_{\{\tilde{S}_T > K\}} \tag{2.13}$$

Now compute the risk-neutral expectation

$$\begin{aligned}
E_t^{\mathbb{Q}}\left[\max(\tilde{S}_T - K, 0)\right] &= E_t^{\mathbb{Q}}[\tilde{S}_T 1_{\{\tilde{S}_T > K\}}] - K E_t^{\mathbb{Q}}[1_{\{\tilde{S}_T > K\}}] && (2.14)\\
&= E_t^{\mathbb{Q}}[\tilde{S}_T] E_t^{\mathbb{Q}}[1_{\{\tilde{S}_T > K\}}] + Cov_t^{\mathbb{Q}}[\tilde{S}_T, 1_{\{\tilde{S}_T > K\}}] \\
&\quad - K E_t^{\mathbb{Q}}[1_{\{\tilde{S}_T > K\}}] && (2.15)\\
&= e^{r(T-t)} \tilde{S}_t E_t^{\mathbb{Q}}[1_{\{\tilde{S}_T > K\}}] + Cov_t^{\mathbb{Q}}[\tilde{S}_T, 1_{\{\tilde{S}_T > K\}}] \\
&\quad - K E_t^{\mathbb{Q}}[1_{\{\tilde{S}_T > K\}}] && (2.16)\\
&= e^{r(T-t)} \tilde{S}_t \left\{ E_t^{\mathbb{Q}}[1_{\{\tilde{S}_T > K\}}] + Cov_t^{\mathbb{Q}}\left[\frac{\tilde{S}_T}{e^{r(T-t)} S_t}, 1_{\{\tilde{S}_T > K\}}\right] \right\} \\
&\quad - K E_t^{\mathbb{Q}}[1_{\{\tilde{S}_T > K\}}] && (2.17)
\end{aligned}$$

$$
\begin{aligned}
E_t^{\mathbb{Q}}\left[\max(\tilde{S}_T - K, 0)\right] &= e^{r(T-t)}\tilde{S}_t\left\{E_t^{\mathbb{Q}}[1_{\{\tilde{S}_T>K\}}] + Cov_t^{\mathbb{Q}}\left[\frac{\tilde{S}_T}{e^{r(T-t)}S_t}, 1_{\{\tilde{S}_T>K\}}\right]\right\} \\
&\quad -KE_t^{\mathbb{Q}}[1_{\{\tilde{S}_T>K\}}] \qquad (2.18) \\
&= e^{r(T-t)}\tilde{S}_t\left\{\mathbb{Q}\{\tilde{S}_T>K\} + Cov_t^{\mathbb{Q}}\left[\frac{\tilde{S}_T}{e^{r(T-t)}S_t}, 1_{\{\tilde{S}_T>K\}}\right]\right\} \\
&\quad -K\mathbb{Q}\{\tilde{S}_T>K\} \qquad (2.19)
\end{aligned}
$$

$$
\begin{aligned}
N(d_{2,t}) &= \overbrace{\mathbb{Q}\{\tilde{S}_T>K\}}^{\text{risk-neutral prob. of being ITM at date } T} \qquad (2.20) \\
N(d_{1,t}) &= \mathbb{Q}\{\tilde{S}_T>K\} + \underbrace{Cov_t^{\mathbb{Q}}\left[\frac{\tilde{S}_T}{e^{r(T-t)}S_t}, 1_{\{\tilde{S}_T>K\}}\right]}_{\text{additional term measuring covariance between stock returns and the event of being ITM at date } T} \qquad (2.21)
\end{aligned}
$$

(2.22)

$$
\begin{aligned}
c_t &= e^{-r(T-t)} E_t^{\mathbb{Q}} \left[\max(\tilde{S}_T - K, 0) \right] && (2.23) \\
&= PV_t[\tilde{S}_T] N(d_{1,t}) - PV_t[K] N(d_{2,t}) && (2.24) \\
&= S_t N(d_{1,t}) - e^{-r(T-t)} K N(d_{2,t}) && (2.25)
\end{aligned}
$$

Further reading: Chapters 2 & 3 of A Course in Derivative Securities: Introduction to Theory and Computation by Kerry Back.

Understanding the Black-Scholes assumption for stock prices.

ln $\frac{\tilde{S}_T}{S_t}$ is the continuously compounded return over the interval $[t, T]$.

If you downloaded daily, monthly or annual data on stock prices, you could compute daily, monthly or annual continuously compounded returns. Plotting a histogram of the continuously compounded returns would reveal that they are close to normal. So it seems that the **physical** probability distribution of continuously compounded stock returns is approximately normal.

The main deviations from normality concern:

- Volatility clustering: large moves in stock returns follow large moves and small moves follow small moves

- Distribution of stock returns is highly peaked and fat-tailed relative to Normal distribution – characteristic of mixtures of distributions with different variances.

Figure 1: SPX daily log returns from 1/1/1990 to 31/12/1999

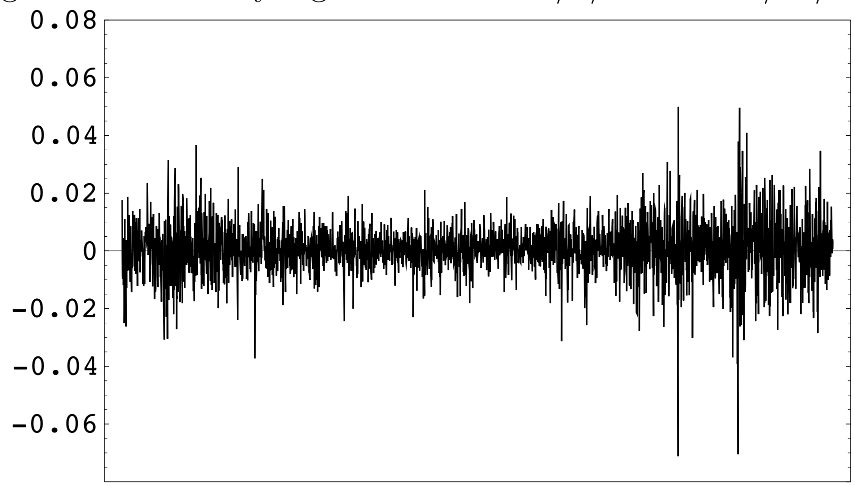


Figure 2: Frequency distribution of SPX daily log returns from 1/1/1990 to 31/12/1999 compared with the Normal distribution

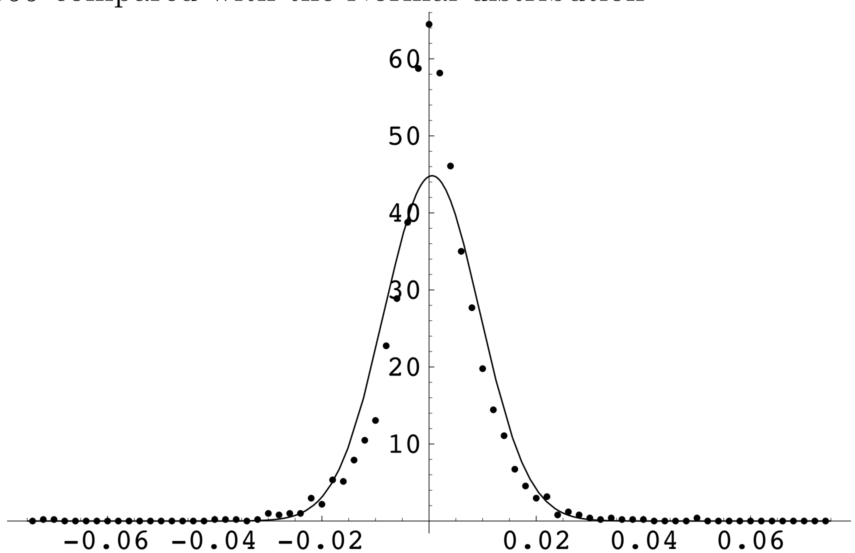
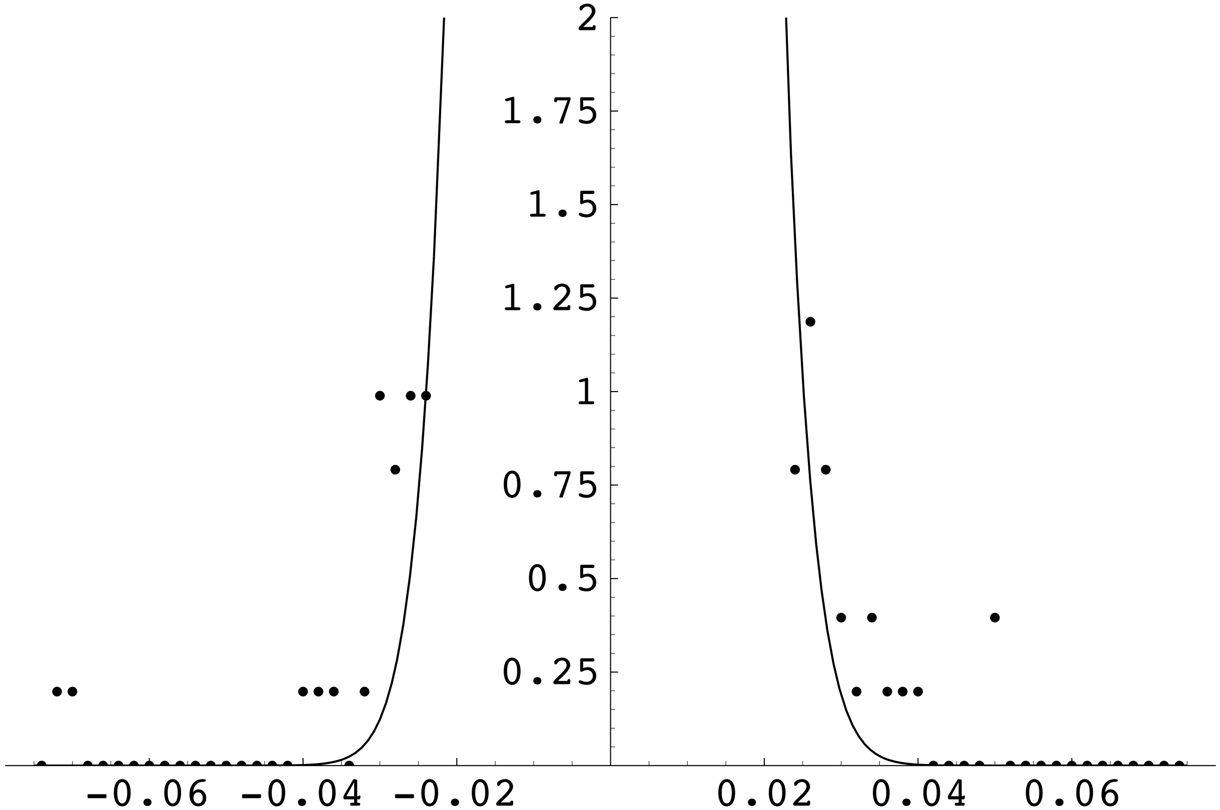


Figure 3: Tails of SPX frequency distribution



Despite its faults, let us proceed with the assumption that the **physical** probability distribution of continuously compounded stock returns is normal.

What about the risk-neutral distribution? Let is assume it is still normal. Its mean will be smaller than for the physical distribution. We also know that certainty equivalents are discounted at the risk-free rate. This means that the expected risk premium for stock returns under the risk-neutral measure is zero. Hence using risk-neutral probabilities, the mean gross return on the stock depends on the risk-free rate, i.e.

$$E_t^{\mathbb{Q}}[\tilde{S}_T] = S_t e^{r(T-t)} \tag{2.26}$$

and

$$E_t\left[\ln \frac{\tilde{S}_T}{S_t}\right] = \left(r - \frac{1}{2}\sigma^2\right)(T-t). \tag{2.27}$$

So it makes sense that with risk-neutral probabilities

$$\ln \frac{\tilde{S}_T}{S_t} \sim N\left[\left(r - \frac{1}{2}\sigma^2\right)(T-t), \sigma\sqrt{T-t}\right] \tag{2.28}$$